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Reflection of a normal incident matter wave by an impenetrable wall moving in a constant velocity is studied. If the wall moves slowly enough, the reflected wave propagates toward the opposite direction of the incident wave. However, if the the wall moves faster than the phase velocity of the incident wave, both the reflected and incident waves propagate in the same direction. Analysis of the probability density and current reveals that this counter-intuitive result satisfies physical requirements and is what indeed happens.

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The exact solution of the time-dependent Schrödinger equation has drawn much attention over the past decades [1–3]. Besides the mathematical interest, these kinds of study may help us to further explore fascinating quantum phenomenon or to clarify some unconventional issue [4,5], which have not yet been discussed in usual textbooks [6,7]. The purpose of this note is to provide a very simple example of one-dimensional quantum scattering problem which reveals some unexpected results. We believe the phenomenon described blew are important enough and deserves more related studies.

Consider a particle of mass m and momentum $p = \hbar k$ is incident from left and be reflected by a wall moving with a constant velocity v. The total matter wave ψ satisfies the Schrödinger equation $i\partial_t \psi(x,t) = -(\hbar^2/2m)\partial_x^2 \psi(x,t)$ (here $x < vt$, and vt is the position of the wall at time t) is the sum of the incident wave $\psi_+ = e^{i(kx-\omega t)}$ and reflected wave $\psi_- = re^{i(k'x-\omega' t)}$:

$$\psi(x,t) = e^{i(kx-\omega t)} + re^{i(k'x-\omega' t)}, \tag{1}$$

here $\omega = \hbar k^2/2m$ and $\omega' = \hbar k'^2/2m$.

By definition a total reflecting wall is the boundary separates the regions of potential $V(x) = 0$ and $V(x) = \infty$ where wave function vanishes. Since the wall moves uniformly with velocity v we have the boundary condition

$$\psi(vt,t) = 0 \tag{2}$$

which leads to the solution

$$r = -1, \qquad k' = -k + \frac{2mv}{\hbar}, \tag{3}$$

and the phase velocity v_p of the reflected wave ψ_- is given by

$$v_p = \frac{\hbar}{2m}(-k + \frac{2mv}{\hbar}) = v - \frac{\hbar k}{2m}. \tag{4}$$

Up to now everything seems reasonable. For a wall moves toward the direction of $-x$ or moves slowly enough we have $k' < 0$ and the reflected wave propagates toward the opposite direction of the incident wave, which is consistent with our naive intuition. However, what will happen if the wall moves fast enough so that $2mv > \hbar k$? Surprisingly, in this situation $k' > 0$ and hence the incident and reflected waves propagate toward the same direction! Furthermore, if we increase the velocity of the wall such that $v > \hbar k/m$ then the reflected wave will propagate faster than the incident wave. One might feel uncomfortable and doubt if these counter-intuitive phenomenon will actually happen. However, since the above derivation based on merely the Schrödinger equation itself and the simple boundary condition (2), we believe they are indeed what will happen under the given conditions.

To further explore the interesting phenomenon, we analyze the problem as follows. Substitute (3) into (1), and define

$$\bar{x} = x - vt, \qquad \bar{k} = k - \frac{mv}{\hbar}, \tag{5}$$

we have

$$\psi(x,t) = \exp\left[i\left(\frac{mv}{\hbar}x - \frac{mv^2}{2\hbar}t\right)\right]\varphi(\bar{x},t), \tag{6}$$

where

$$\varphi(\bar{x},t) = 2i\sin(\bar{k}\bar{x})\exp\left(-i\frac{\hbar\bar{k}^2}{2m}t\right). \tag{7}$$

Note that Eq.(6) is nothing but the Galilean Transformation to an inertial frame moving with velocity v, and $\varphi(\bar{x},t)$ is the wave function in that system.

From these results we have the unnormalized probability density

$$|\psi|^2 = 4\sin^2\left[\left(k - \frac{mv}{\hbar}\right)(x - vt)\right] \tag{8}$$

and the probability current

$$\begin{aligned} J &= (\hbar/2mi)(\psi^*\partial_x\psi - \psi\partial_x\psi^*) \\ &= 4v\sin^2\left[\left(k - \frac{mv}{\hbar}\right)(x - vt)\right]. \end{aligned} \tag{9}$$

Now we define a "drift velocity" v_d as the ratio $J/|\psi|^2$, then we find the reasonable result

$$v_d = v, \tag{10}$$

which means the fixed pattern of the infinite train of the particle probability density $|\psi|^2$ behind the wall is dragged by the wall and moves uniformly with velocity v.

In conclusion, we have shown that under appropriate conditions even the simplest one-dimensional quantum scattering shows unexpected results. We believe the phenomenon described in this note are important enough and hope it can stimulate more related studies.

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